INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
NOVEMBER 2024

FURTHER STUDIES MATHEMATICS (STANDARD): PAPER I

MARKING GUIDELINES

Time: 2 hours 200 marks

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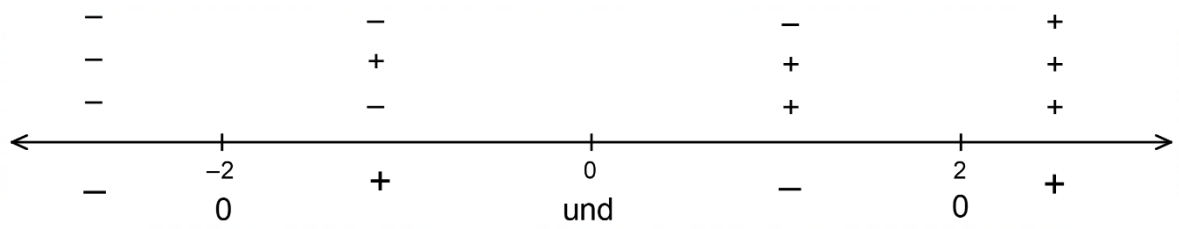
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QUESTION 1

1.1 Solve for $x \in \mathbb{R}$:

(a) $x \leq \frac{4}{x}$

$$x - \frac{4}{x} \leq 0$$
$$\frac{x^2 - 4}{x} \leq 0$$
$$\frac{(x-2)(x+2)}{x} \leq 0$$



$x \leq -2$ or $0 < x \leq 2$

(b) $\ln(fe^x + c) = k$

$$e^k = fe^x + c$$
$$e^k - c = fe^x$$
$$\frac{e^k - c}{f} = e^x$$
$$x = \ln\left(\frac{e^k - c}{f}\right)$$

1.2 Solve for $x \in \mathbb{R}$

$$\left|x^2 + 6x\right| = 8$$

$x^2 + 6x = 8$ or $x^2 + 6x = -8$

$x^2 + 6x - 8 = 0$ or $x^2 + 6x + 8 = 0$

$x = \frac{-6 \pm \sqrt{68}}{2}$ or $(x+2)(x+4) = 0$

$x = -3 \pm \sqrt{17}$ or $x = -2$ or $x = -4$

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1.3 One of the factors of $f(x) = 2x^5 + 5x^4 + 13x^3 + 15x^2 - 2x - 3$ is $(2x + 3)$. Find the corresponding factor.

Method 1 – by inspection:

$$2x^5 + 5x^4 + 13x^3 + 15x^2 - 2x - 3$$
$$= (2x + 3)(x^4 + x^3 + 5x^2 - 1)$$

Method 2 – by long division:

$$2x + 3 \overline{)\, 2x^5 + 5x^4 + 13x^3 + 15x^2 - 2x - 3}$$

$$x^4 + x^3 + 5x^2 - 1$$

$$\begin{array}{r} 2x^5 + 3x^4 \\ \hline 2x^4 + 13x^3 \\ 2x^4 + 3x^3 \\ \hline 10x^3 + 15x^2 \\ 10x^3 + 15x^2 \\ \hline 0 \quad -2x - 3 \\ -2x - 3 \\ \hline 0 \end{array}$$

1.4 Determine real values of *a* and *b* if:

$$(a + bi)(4 - i) = a + 3bi + 3b + i$$

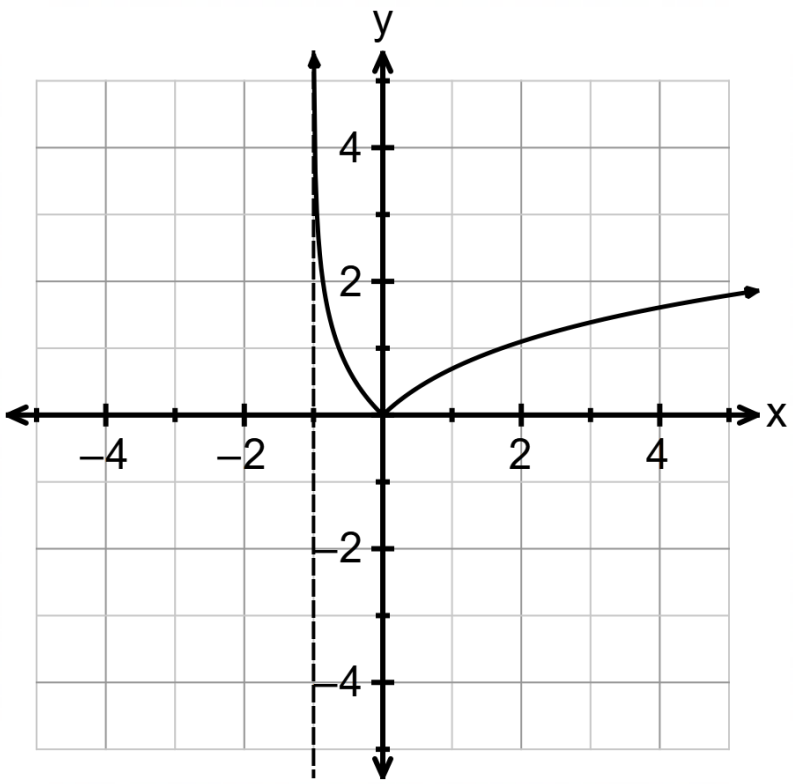
$$4a - ai + 4bi - bi^2 = (a + 3b) + (3b + 1)i$$
$$(4a + b) + (4b - a)i = (a + 3b) + (3b + 1)i$$
$$\therefore 4a + b = a + 3b \text{ ① and ② } 4b - a = 3b + 1$$
from ② $a = b - 1$

Substituting this in ① gives:
$$4(b - 1) + b = b - 1 + 3b$$
$$5b - 4 = 4b - 1$$
$$b = 3 \text{ and } a = 2$$

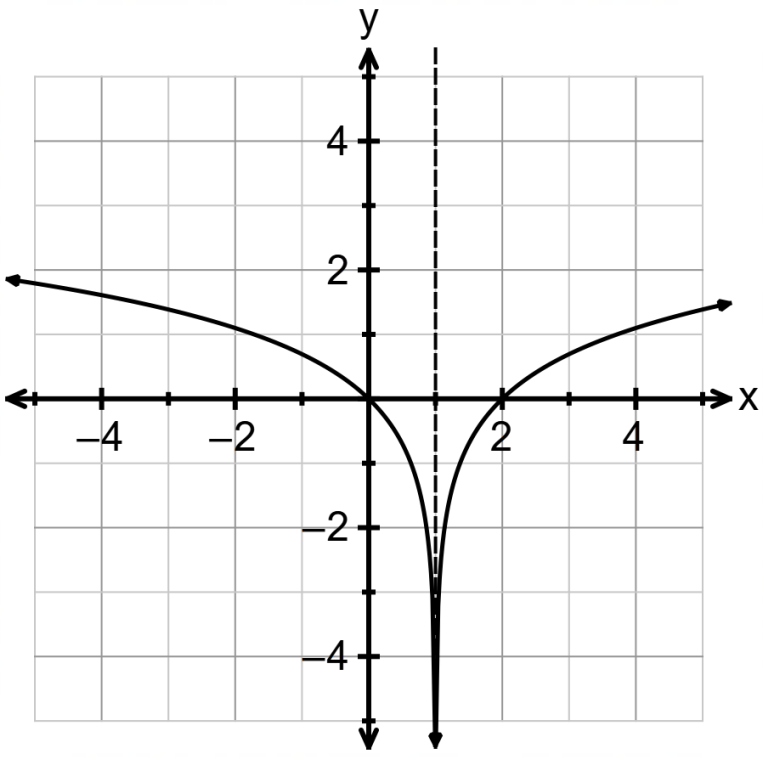
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1.5 Sketch the following functions on the axes provided. Also indicate the equations of any asymptotes and show any intercepts with the axes.

(a) $y = \left|\ln(x+1)\right|$



(b) $y = \ln\left|x-1\right|$



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QUESTION 2

Use mathematical induction to prove that:

$6^n - 1$ is divisible by 5 for all $n \in \mathbb{N}$

If $n = 1$, we have $6^1 - 1 = 5$
So, it is true for $n = 1$

Now assume true for $n = k$

$6^k - 1 = 5p$ where $p \in \mathbb{N}$ (*)

Now consider when $n = k + 1$

$6^{k+1} - 1$
$= 6(6^k) - 1$ ①

From (*) we know that $6^k = 5p + 1$

Substituting this into ① gives:
$6(5p + 1) - 1$
$= 30p + 5$
$= 5(6p + 1)$

So, it is true for $n = k + 1$

Therefore, by the principle of mathematical induction
$6^n - 1$ is divisible by 5 for $n \in \mathbb{N}$.

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QUESTION 3

If $f(x)=\frac{1}{\sqrt{x}}$ then determine $f'(x)$ by first principle.

$$f'(x)=\lim_{h\to 0}\frac{\left(\frac{1}{\sqrt{x+h}}\right)-\left(\frac{1}{\sqrt{x}}\right)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{\sqrt{x}}{\sqrt{x}\sqrt{x+h}}-\frac{\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\times\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{-h}{\left(\sqrt{x}\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{-1}{\left(\sqrt{x}\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$f'(x)=\lim_{h\to 0}\frac{-1}{x\sqrt{x+h}+(x+h)\left(\sqrt{x}\right)}$$

$$f'(x)=\frac{-1}{2x\sqrt{x}} \text{ or } -\frac{1}{2x^{\frac{3}{2}}}$$

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QUESTION 4

4.1 Find, with classification, all asymptotes of the following rational functions. You should give the value(s) of x at which any removable discontinuities exist.

(a) $y = \dfrac{2x^2 + 6}{x^2 - 4}$

$$y = \frac{2x^2 + 6}{(x-2)(x+2)}$$

horizontal asymptote: $y = 2$
vertical asymptotes: $x = 2$ and $x = -2$

(b) $y = \dfrac{x^2 + 4x + 3}{2x + 1}$

$$x^2 + 4x + 3 = (2x+1)\left(\frac{1}{2}x + \frac{7}{4}\right) + \frac{5}{4}$$

$$\therefore \frac{x^2 + 4x + 3}{2x + 1} = \frac{1}{2}x + \frac{7}{4} + \frac{5}{4(2x+1)}$$

$\therefore y = \dfrac{1}{2}x + \dfrac{7}{4}$ is an oblique asymptote
$x = -\dfrac{1}{2}$ is a vertical asymptote

(c) $y = \dfrac{x^2 - 6x + 8}{x^3 - 2x^2 - 8x}$

$$y = \frac{(x-4)(x-2)}{x(x^2 - 2x - 8)}$$

$$y = \frac{(x-4)(x-2)}{x(x-4)(x+2)}$$

horizontal asymptote: $y = 0$
vertical asymptote: $x = -2$ and $x = 0$
removable discontinuity: $x = 4$

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4.2 Consider the function: $f(x) = \dfrac{x^2 + 2x - 5}{2x - 3}$

(a) Determine the coordinates of its stationary points.

$$f(x) = \frac{x^2 + 2x - 5}{2x - 3}$$

$$f'(x) = \frac{(2x + 2)(2x - 3) - 2(x^2 + 2x - 5)}{(2x - 3)^2} = 0$$

$$4x^2 - 2x - 6 - 2x^2 - 4x + 10 = 0$$

$$2x^2 - 6x + 4 = 0$$

$$x^2 - 3x + 2 = 0$$

$$(x - 2)(x - 1) = 0$$

$x = 2$ or 1
(2 ; 3) and (1 ; 2) are the stationary points

(b) Use the fact that $f''(x) = \dfrac{4x - 6}{(2x - 3)^4}$ to justify the nature of the stationary points you found in Question 4.2 (a).

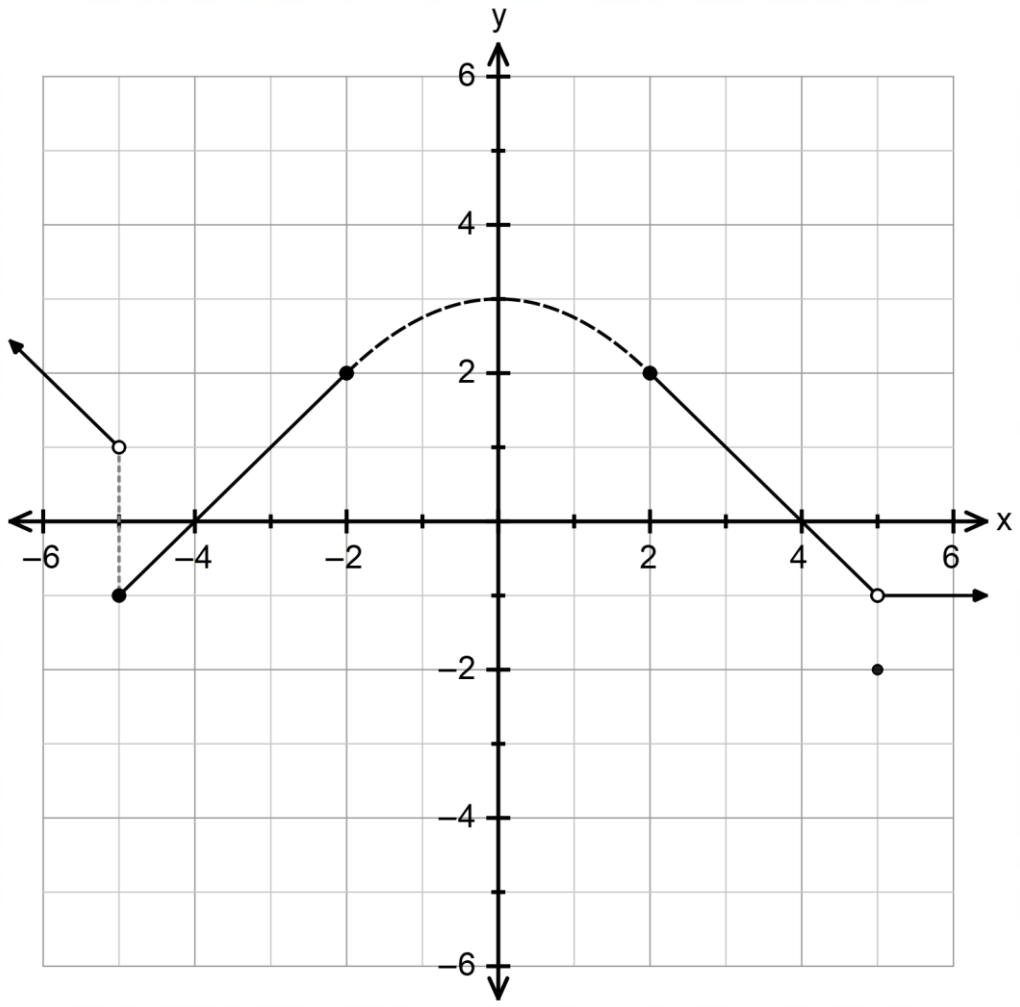
$f''(1) < 0$ so (1 ; 2) is a local maximum

$f''(2) > 0$ so (2 ; 3) is a local minimum

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QUESTION 5

Consider the function f shown below:



(a) Write a mathematical statement to justify why f is discontinuous at:

(1) $x = -5$

$\lim\limits_{x \to -5} f(x)$ d.n.e since $\lim\limits_{x \to -5^-} f(x) \neq \lim\limits_{x \to -5^+} f(x)$

(2) $x = 5$

$\lim\limits_{x \to 5} f(x) \neq f(5)$

(b) Classify the discontinuity that exists at:

(1) $x = -5$

non-removable **OR** jump

(2) $x = 5$

removable

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(c) Suppose $g(x) = \begin{cases} f(x) & -4 \le x < -2 \\ ax^2 + bx + c & -2 \le x < 2 \\ f(x) & 2 \le x \le 4 \end{cases}$

Find *a, b* and *c* if $g(x)$ is differentiable at every point on the interval $[-4\,;\,4]$.

It must be symmetrical about the *y*-axis

so it will be of the form $g(x) = ax^2 + p$

For continuity we know that $g(2) = g(-2) = 2$

so $4a + p = 2$ ①

For differentiability we know that

$\lim\limits_{x \to 2} f'(x) = -1$

so $\lim\limits_{x \to 2} (2ax) = -1$

$\therefore 4a = -1$

$\therefore a = -\frac{1}{4}$ and $p = 3$

$\therefore g(x) = -\frac{1}{4}x^2 + 3$ for $x \in [-2\,;\,2)$

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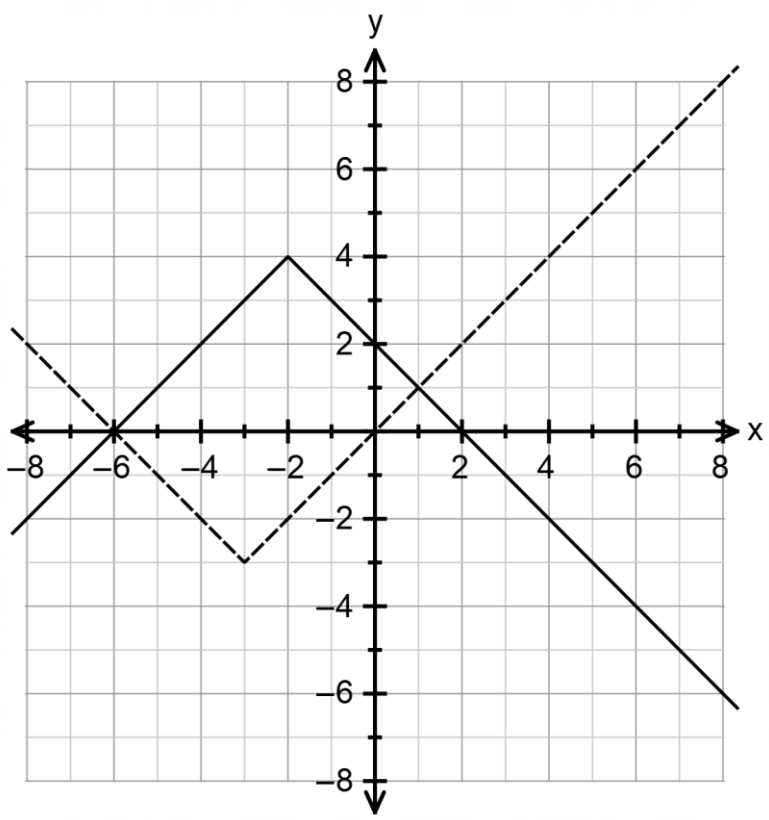
QUESTION 6

6.1 The function $y = -|x+2|+4$ has been drawn below.

Draw another function which will enable you to solve:

$|x+2| \geq -|x+3|+7$ graphically.

Hence, or otherwise, give the solution to the inequality.



$|x+2| \geq -|x+3|+7$

$-|x+2| \leq |x+3|-7$

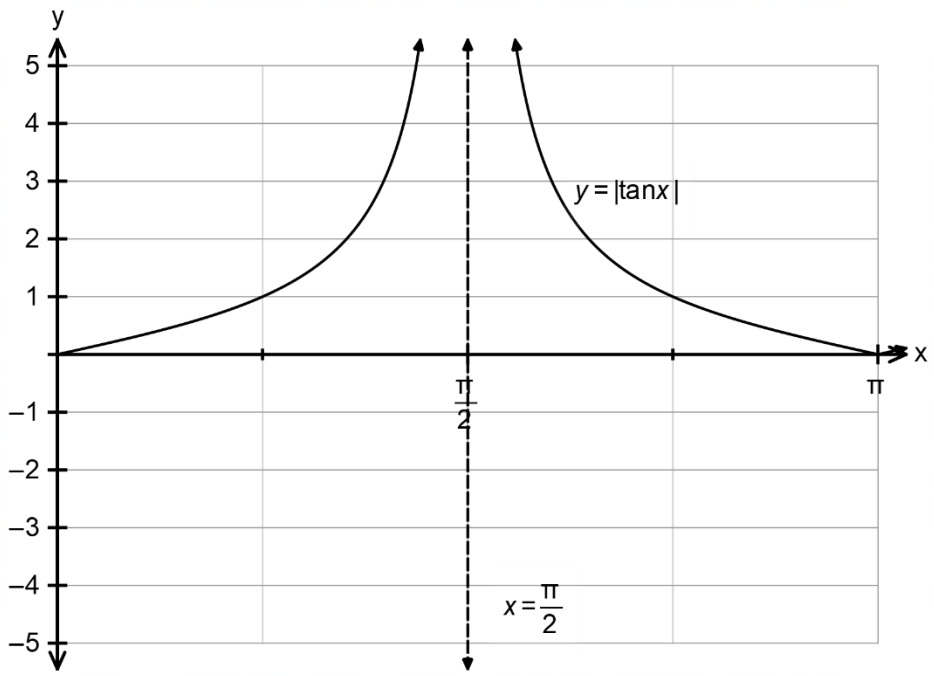
$-|x+2|+4 \leq |x+3|-3$

So, we should draw $y = |x+3|-3$ (see above)
Solution is $x \leq -6$ or $x \geq 1$

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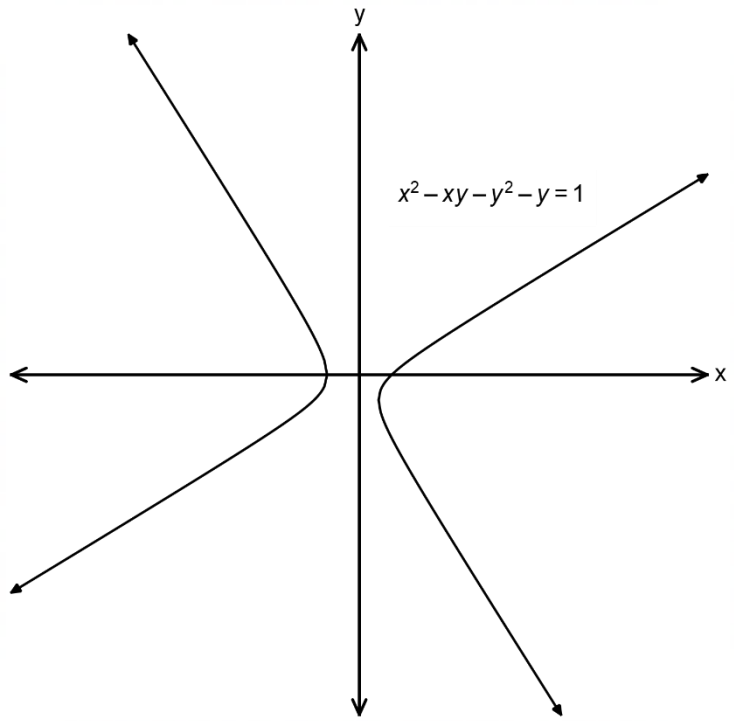
6.2 Draw the graph of $y = \left| \tan x \right|$ on the axes provided showing all points of interest.

You should also draw and label any asymptotes with their equations.



QUESTION 7

A portion of the graph of the implicitly defined relationship $x^2 - xy - y^2 - y = 1$ is shown below.



(a) Determine the coordinates of the positive x-intercept.

Let $y = 0$

$$x^2 = 1$$

$$x = \pm 1$$

so (1 ; 0)

(b) Find the equation of the tangent to the curve at the point found in Question 7 (a).

Differentiating implicitly gives:

$$2x - y - x\frac{dy}{dx} - 2y\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(-x - 2y - 1) = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{-x - 2y - 1}$$

Substitute (1 ; 0)

$$\frac{dy}{dx} = \frac{-2}{-2} = 1$$

$$0 = 1(1) + c$$

$$y = x - 1$$

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QUESTION 8

(a) You have learnt how to differentiate e^x and ln$x$ but what about differentiating exponential and log functions of other bases?

Given $a^x = e^{\ln a^x} = e^{x\ln a}$ and $\log_a x = \dfrac{\ln x}{\ln a}$

Hence, determine:

$$\frac{d}{dx}(2^x + \log_3 x)$$

$$= \frac{d}{dx}\left(e^{x\ln 2} + \frac{\ln x}{\ln 3}\right)$$

$$= e^{x\ln 2}\ln 2 + \frac{1}{x\ln 3} \quad \textbf{OR} \quad 2^x \ln 2 + \frac{1}{x\ln 3}$$

(b) Solve $\cos^2\left(\theta - \dfrac{\pi}{6}\right) - \sin^2\left(\theta - \dfrac{\pi}{6}\right) = -\dfrac{1}{2}$ for $\theta \in [0\,;\,2\pi]$.

$$\cos\left(2\theta - \frac{\pi}{3}\right) = -\frac{1}{2}$$

key $\angle = \dfrac{\pi}{3}$ in quadrants 2 and 3

$$2\theta - \frac{\pi}{3} = \pi - \frac{\pi}{3} + 2\pi k \quad \text{or} \quad 2\theta - \frac{\pi}{3} = \pi + \frac{\pi}{3} + 2\pi k \text{ both with } k \in \mathbb{Z}$$

$$2\theta = \pi + 2\pi k \quad \text{or} \quad 2\theta = \frac{5\pi}{3} + 2\pi k$$

$$\theta = \frac{\pi}{2} + \pi k \quad \text{or} \quad \theta = \frac{5\pi}{6} + \pi k$$

$$\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{6} \text{ or } \frac{11\pi}{6}$$

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QUESTION 9

Determine the following integrals:

(a) $\int x\sec^2(5x^2)\,dx$

Method 1 – by inspection

$\int x\sec^2(5x^2)\,dx$

$= \frac{\tan(5x^2)}{10} + c$

Method 2 – by substitution

$\int x\sec^2(5x^2)\,dx$

let $u = 5x^2$ then $\frac{du}{dx} = 10x$ and $dx = \frac{du}{10x}$

$= \frac{1}{10}\int \sec^2 u\,du$

$= \frac{1}{10}\tan u + c$

$= \frac{1}{10}\tan(5x^2) + c$

ALTERNATIVE MEMO FOR THOSE USING INCORRECT FORMULA AS PROVIDED IN INFORMATION BOOKLET

Method 1 – by inspection

$\int x\sec^2(5x)\,dx$

$= \frac{\cot(5x^2)}{10} + c$

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Method 2 – by substitution

$$\int x\sec^2(5x)\ dx$$

let $u = 5x^2$ then $\frac{du}{dx} = 10x$ and $dx = \frac{du}{10x}$

$$= \frac{1}{10}\int \sec^2 u\ \text{du}$$

$$= \frac{1}{10}\cot u + c$$

$$= \frac{1}{10}\cot(5x^2) + c$$

(b) $\int 3 + \cot^2 x\ dx$

$$= \int 2 + 1 + \cot^2 x\ dx$$

$$= \int 2 + \text{cosec}^2 x\ dx$$

$$= 2x - \cot x + c$$

(c) $\int \frac{x^3 + 4}{x^2 - 4}\ dx$

$$x^3 + 4 = (x^2 - 4)x + 4x + 4$$

$$\frac{x^3 + 4}{x^2 - 4} = x + \frac{4x + 4}{(x - 2)(x + 2)}$$

now $\frac{4x + 4}{(x - 2)(x + 2)} = \frac{A}{x - 2} + \frac{B}{x + 2}$

cover up method gives $A = 3$ and $B = 1$

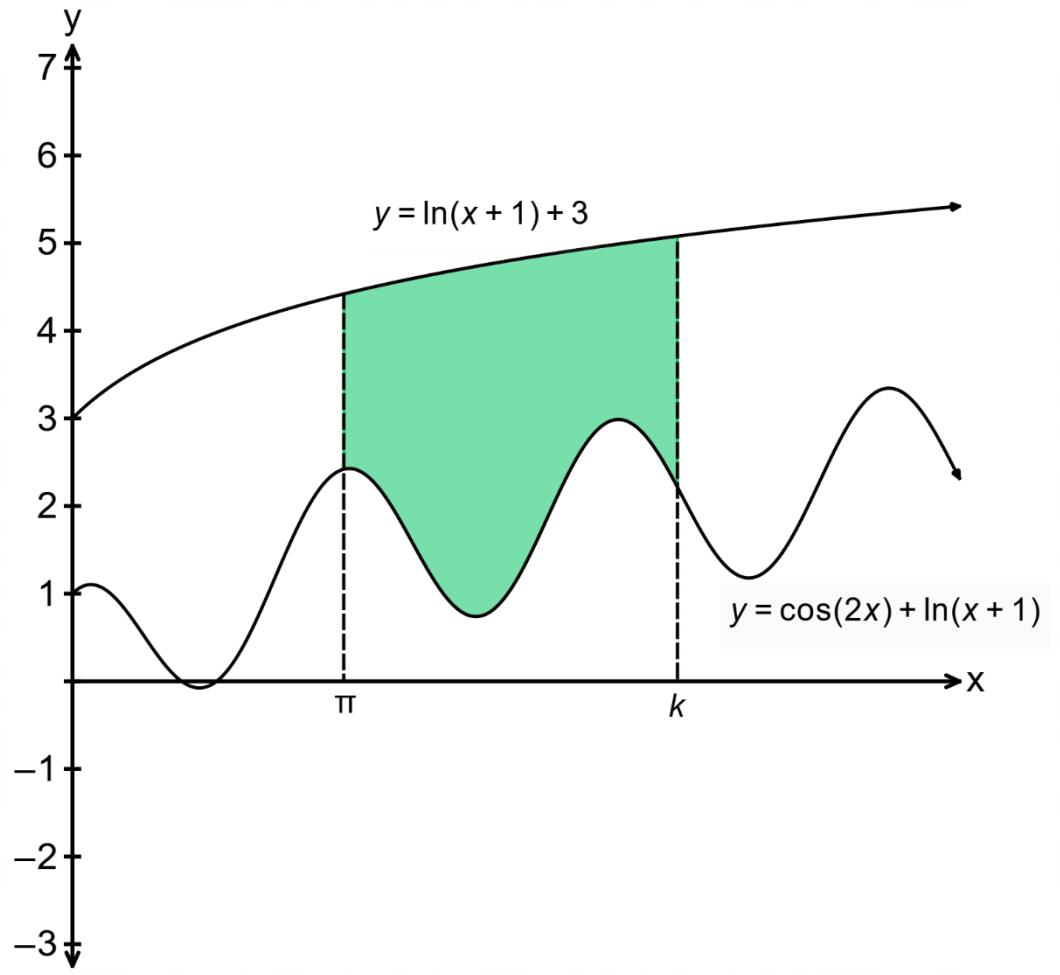
$$\therefore \int \frac{x^3 + 4}{x^2 - 4}\ dx = \int x + \frac{3}{x - 2} + \frac{1}{x + 2}\ dx$$

$$= \frac{x^2}{2} + 3\ln|x - 2| + \ln|x + 2| + c$$

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QUESTION 10

(a) Determine the shaded area in terms of k showing all working details. You should evaluate any integrals in your answer.



$$A = \int_{\pi}^{k} \ln(x+1) + 3 - (\cos(2x) + \ln(x+1))\, dx$$

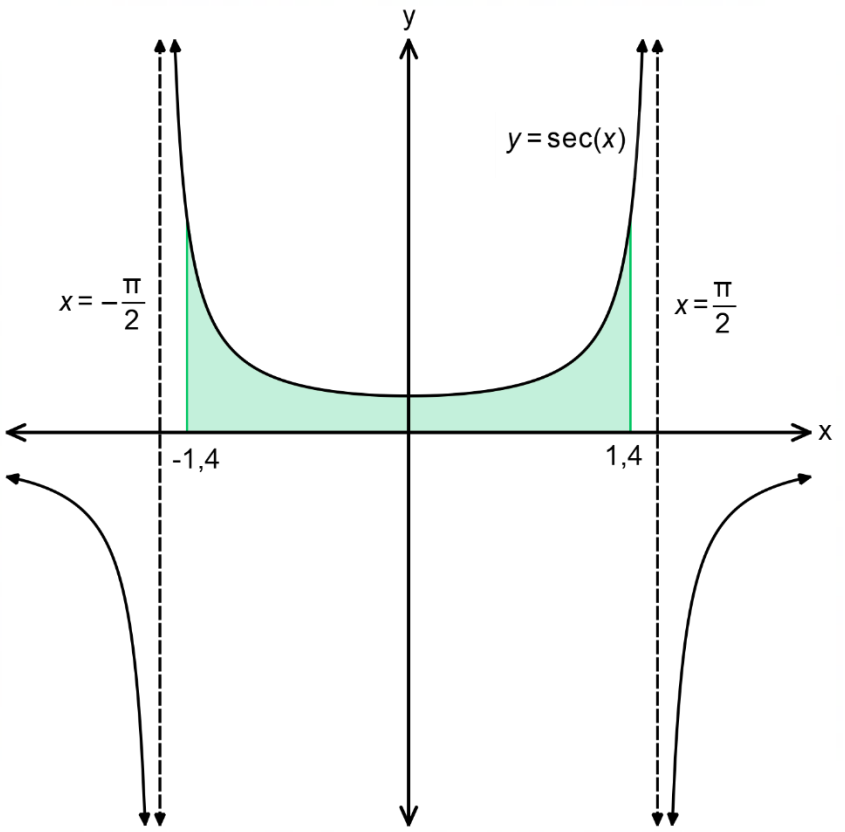
$$A = \int_{\pi}^{k} 3 - \cos(2x)\, dx$$

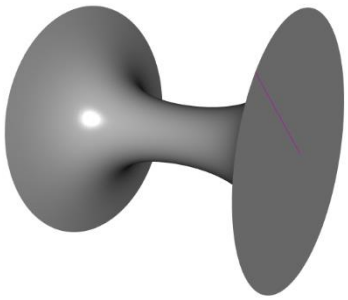
$$A = \left[3x - \frac{\sin(2x)}{2} \right]_{\pi}^{k}$$

$$A = 3k - \frac{\sin(2k)}{2} - \left(3\pi - \frac{\sin(2\pi)}{2} \right)$$

$$A = 3k - \frac{\sin(2k)}{2} - 3\pi$$

(b) Determine, to 1 decimal place, the volume when the shaded area below is rotated about the *x*-axis.





$$vol = \pi\int_{-1,4}^{1,4} \sec^2 x\ dx$$

$$= \pi\left[\tan(1,4) - \tan(-1,4)\right]$$

$$= 36,4 \text{ units}^3$$

ALTERNATIVE MEMO FOR THOSE USING INCORRECT FORMULA AS PROVIDED IN INFORMATION BOOKLET

$$vol = \int_{-1,4}^{1,4} \sec^2 x\ dx$$

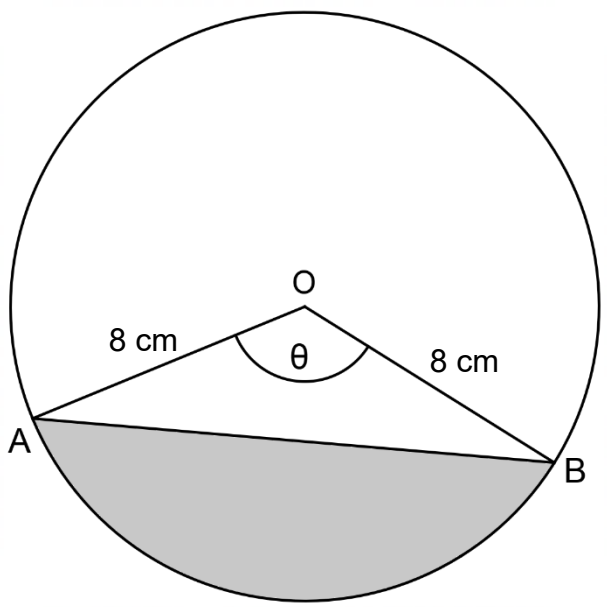
$$= \pi[\cot(1,4) - \cot(-1,4)]$$

$$= 1,1\ units^3$$

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QUESTION 11

In the diagram below, O is the centre of the circle. $A\hat{O}B = \theta$ and $AO = BO = 8$ cm.
The shaded area is 45 cm^2.



Use the Newton-Raphson method to determine θ (in radians) to 5 decimal places.

You should:

- Show the equation you are solving.
- Show the iterative formula you will use.
- Use an initial approximation, $\theta_0 = 1$.
- Give θ_1 to 5 decimal places.

Shaded area $=$ sector – triangle

$$= \frac{1}{2}(8^2)\theta - \frac{1}{2}(8^2)\sin\theta$$
$$= 32\theta - 32\sin\theta$$

$$\therefore 32\theta - 32\sin\theta = 45$$
$$\therefore 32\theta - 32\sin\theta - 45 = 0$$
$$\therefore f(\theta) = 0$$

$$\therefore \theta_{n+1} = \theta - \frac{32\theta - 32\sin\theta - 45}{32 - 32\cos\theta}$$
$$\therefore \theta_1 = 3{,}71422$$
$$\theta = 2{,}20926$$

Total: 200 marks

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